Recall $\begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$

Critical point (x_0, y_0) s.t. $\begin{cases} F(x_0,y_0)=0 \\ G(x_0,y_0)=0 \end{cases}$

Near each critical point, the nonlinear system can be approximated by the linear system: $\vec{u}' = J(x_0,y_0)\vec{u}$

where $\vec{u} = \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$.

$$\begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix} = \begin{bmatrix} F(x_0,y_0) \\ G(x_0,y_0) \end{bmatrix} + \begin{bmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} + O\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right)$$

$$= \begin{bmatrix} F(x_0,y_0) \\ G(x_0,y_0) \end{bmatrix} + \begin{bmatrix} F_x(x_0,y_0)+\varepsilon_{11}(x-x_0,y-y_0) & F_y(x_0,y_0)+\varepsilon_{12}(x-x_0,y-y_0) \\ G_x(x_0,y_0)+\varepsilon_{21}(x-x_0,y-y_0) & G_y(x_0,y_0)+\varepsilon_{22}(x-x_0,y-y_0) \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$$

All ε_{ij}'s are small enough.

For linear approximation, ε_{ij}'s can be omitted.

It will be used in error analysis later.

$$\begin{bmatrix} (x-x_0)' \\ (y-y_0)' \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$$

Lecture #23 July 13, 2015

Example:

$$\begin{Bmatrix} dx/dt = -(x-y)(1-x-y) \\ dy/dt = x(2+y) \end{Bmatrix}$$

Step #1: Critical Points

$$\begin{cases} -(x-y)(1-x-y) = 0 \\ x(2+y) = 0 \end{cases}$$

2^{nd} equation $\Rightarrow$ Either $x = 0$ or $y = -2$

If $x = 0$, 1^{st} equation $\Rightarrow -(0-y)(1-0-y) = 0$

$\Rightarrow y(1-y) = 0$

$\Rightarrow y = 0$ or $y = 1$

If $y = -2$, 1^{st} equation $\Rightarrow -(x+2)(1-x+2) = 0$

$\Rightarrow -(x+2)(3-x) = 0$

$\Rightarrow x = -2$, or $x = 3$

Critical Points: $(0,0)\ (0,1)\ (-2,-2),\ (3,-2)$

Step #2: For each critical point, formulate the linear approximate

$F = -(x-y)(1-x-y)$, $F_x = -(1-x-y) + (x-y) = 1 + 2x$

$F_y = (1-x-y) + (x-y) = 1 - 2y$

$G = x(2+y)$, $G_x = 2+y$, $G_y = x$

$$J(x,y) = \begin{bmatrix} -1+2x & 1-2y \\ 2+y & x \end{bmatrix}$$

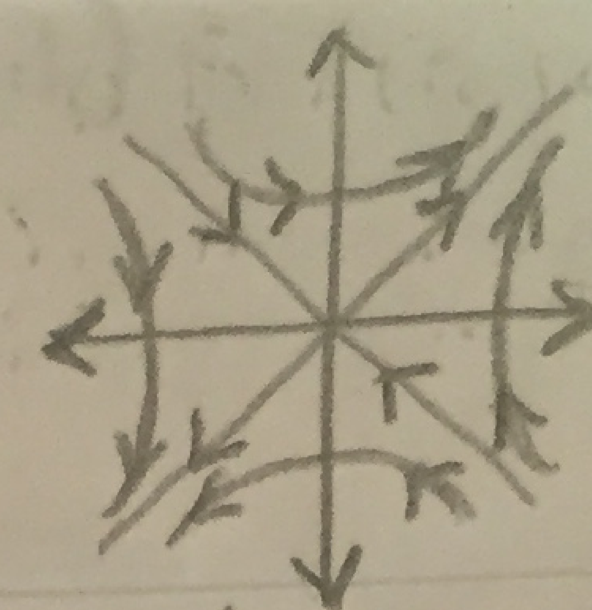
Near $(0,0)$, $\vec{u} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix} \vec{u}$

Eigenvalues

$\Rightarrow$ Eigenvalues: $\begin{vmatrix} -1-\lambda & 1 \\ 2 & 0-\lambda \end{vmatrix} = \lambda^2 + \lambda - 2 = 0$ $\boxed{\begin{matrix} \lambda_1 = -2 \\ \lambda_2 = 1 \end{matrix}}$ $\rightarrow$

Eigenvectors:

$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



• Near $(0,1)$ $\vec{u}' = \begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix} \vec{u} \Rightarrow \begin{vmatrix} -1-\lambda & -1 \\ 3 & -\lambda \end{vmatrix} = \lambda^2 + \lambda + 3 = 0$

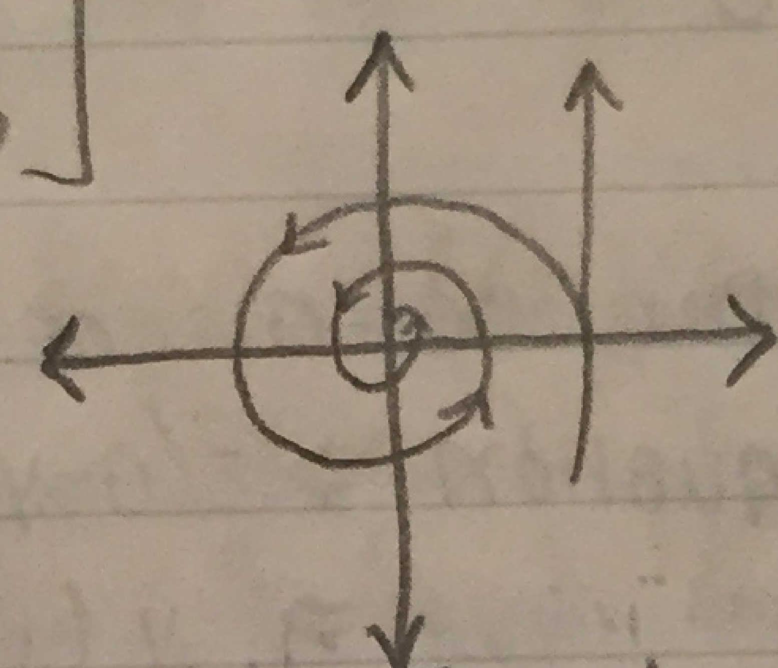
$\lambda_{1,2} = \frac{-1 \pm \sqrt{1^2 - 12}}{2}$

$\lambda_{1,2} = -\frac{1}{2} + i\frac{\sqrt{11}}{2}$

Eigen values

$\Rightarrow$ tangent vector at $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is

$\begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$



spiral sink

the portrait should be moved to (0,1)

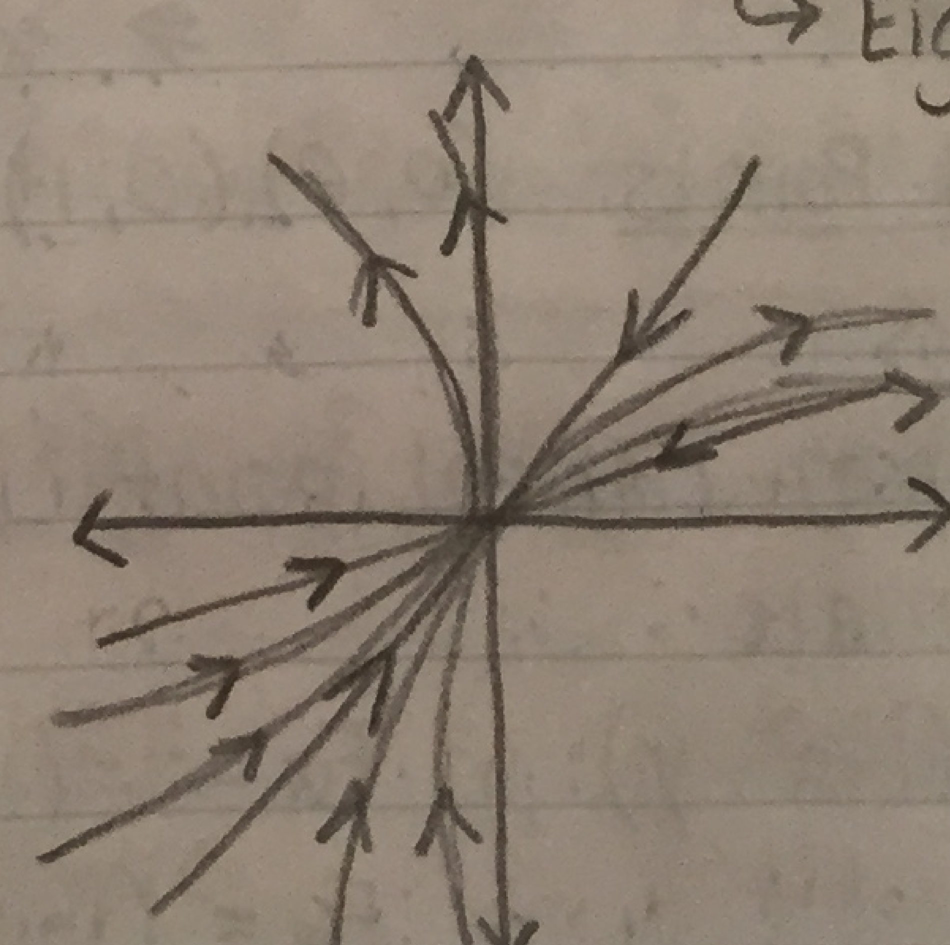
• Near $(-2,-2)$, $\vec{u}' = \begin{bmatrix} -5 & 5 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{vmatrix} -5-\lambda & 5 \\ 0 & -\lambda \end{vmatrix} = (-5-\lambda)(-2-\lambda) = 0$

$\lambda_1 = -5$, $\lambda_2 = -2$

Eigenvalues

Eigenvectors:

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{v}_2 \begin{bmatrix} 5 \\ 3 \end{bmatrix}$



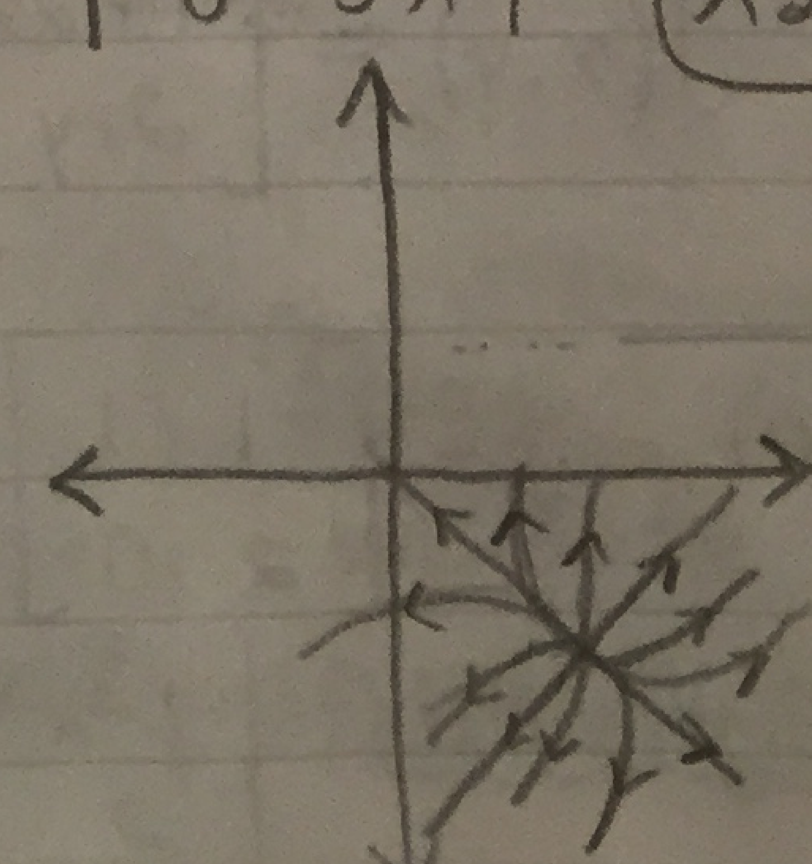
the portrait should be moved to (-2,-2)

• Near $(3,-2)$ $\vec{u}' = \begin{bmatrix} 5 & 5 \\ 0 & 3 \end{bmatrix} \vec{u} \Rightarrow \begin{vmatrix} 5-\lambda & 5 \\ 0 & 3-\lambda \end{vmatrix} \Rightarrow$ $\lambda_1 = 5$, $\lambda_2 = 3$

Eigen-Values

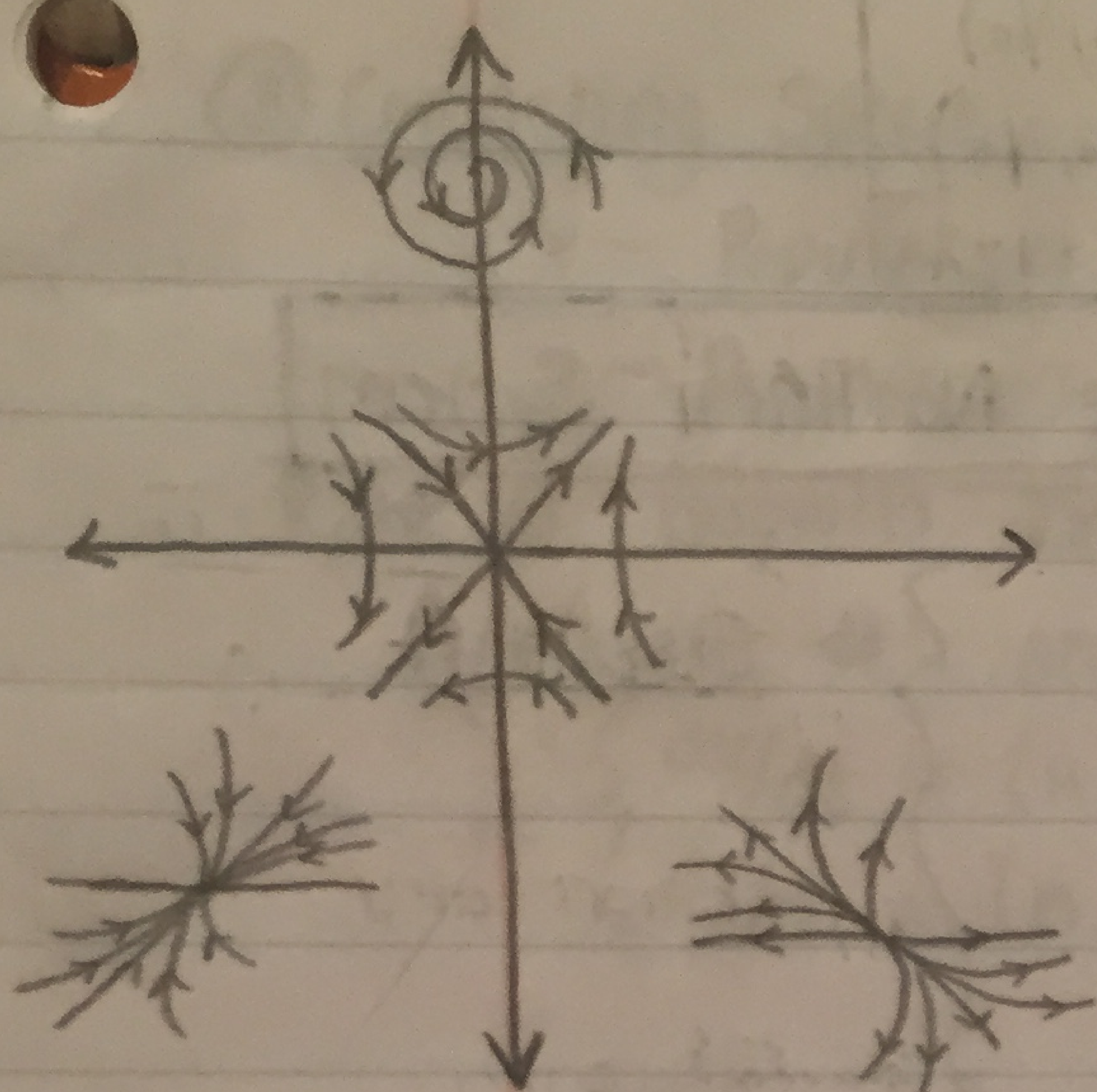
Eigenvectors:

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$



• **Step # 3: Plot Global Phase Portrait**

↳ This would involve some complex computations that would not be required on an Exam!



• **Trace Determinant Classification:**

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = \lambda^2 - \underbrace{(a_{11}+a_{22})}_{TrA}\lambda + \underbrace{(a_{11}a_{22}-a_{12}a_{21})}_{Det\,A} = 0$$

<u>Vieta's Theorem</u>: $\left\{ \begin{array}{l} \lambda_1 + \lambda_2 = trA \\ \lambda_1 - \lambda_2 = det\,A \end{array} \right\}$

<u>Note</u>: If λ_1, λ_2 real #'s, $\lambda_1, \lambda_2 > 0$, then $\lambda_1 + \lambda_2 > 0$

$\Rightarrow$ both λ_1 and $\lambda_2 > 0$

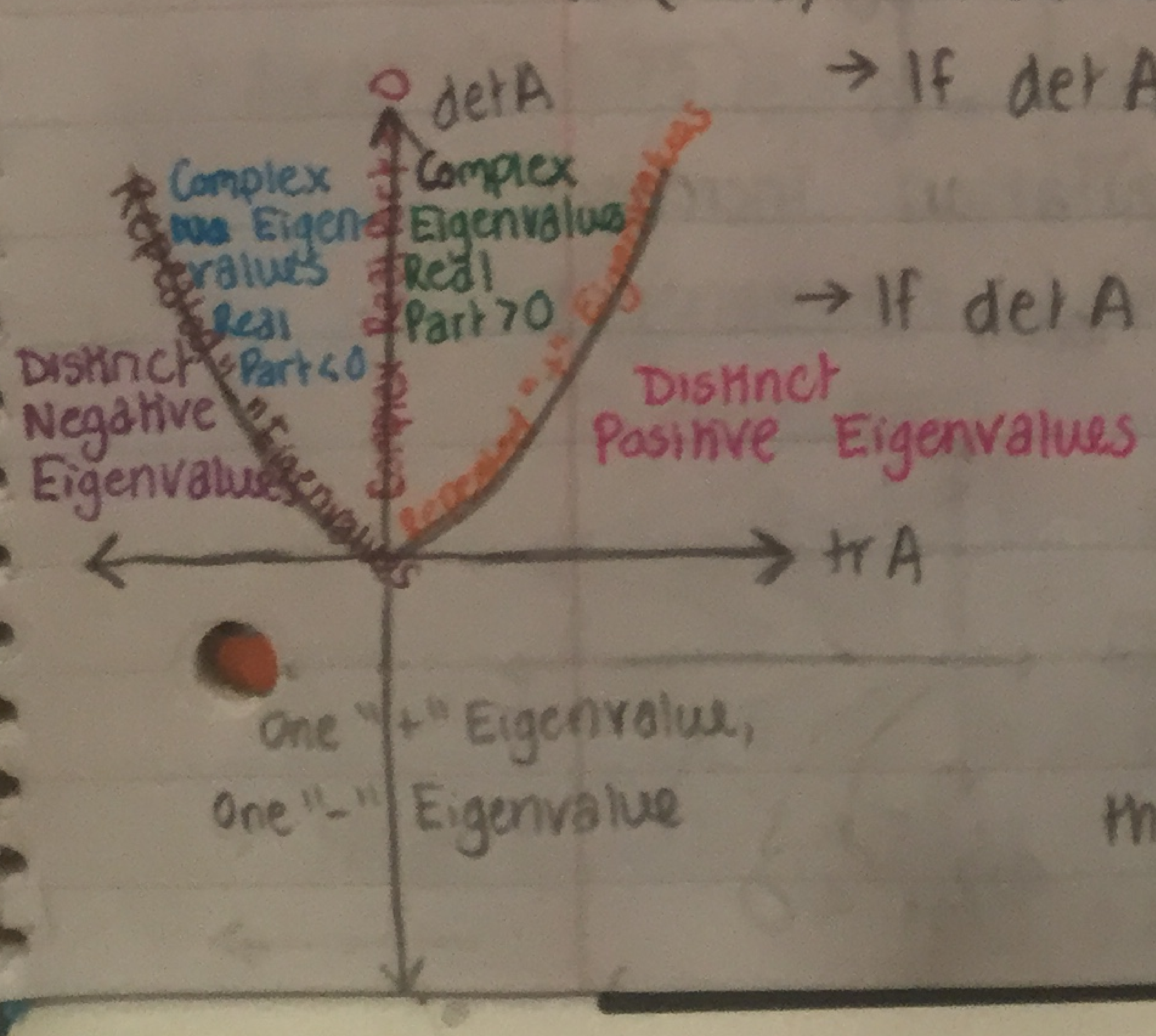
or $\lambda_1 + \lambda_2 < 0 \Rightarrow$ both λ_1 and $\lambda_2 < 0$

If $(trA)^2 - 4\,detA > 0$...

→ If $det A > 0$, then ① $trA > 0 \Rightarrow \lambda_1, \lambda_2 > 0$

② $trA < 0 \Rightarrow \lambda_1, \lambda_2 < 0$

→ If $detA < 0$, then $\lambda_1 > 0$, $\lambda_2 < 0$



*Recall: $\vec{u}' = J(x_0, y_0)\vec{u}$ is an approximat[ion] to the nonlinear system

Nonlinear term acts as perturbations to the x-axis

→

$$\begin{bmatrix} F_x = (x_0, y_0) + \varepsilon_{11}(x - x_0, y - y_0), & F_y = (x_0, y_0) + \varepsilon_{12}(x - x_0, y - y_0) \\ G_x = (x_0, y_0) + \varepsilon_{21}(x - x_0, y - y_0), & G_y = (x_0, y_0) + \varepsilon_{22}(x - x_0, y - y_0) \end{bmatrix}$$

$$\Rightarrow J(x_0, y_0) = \begin{bmatrix} F_x(x_0, y_0), & F_y(x_0, y_0) \\ G_x(x_0, y_0), & G_y(x_0, y_0) \end{bmatrix}$$

Qualitative Properties of the Nonlinear System

Eigenvalues:	Classification:	Stability:
$\lambda_1 > \lambda_2 > 0$		
$\lambda_1 > 0 > \lambda_2$		
$0 > \lambda_1 > \lambda_2$		
$\lambda = \alpha \pm i\beta, \ \alpha > 0$		
* $\lambda = \alpha \pm i\beta, \ \alpha = 0$	Indefinite	Indefinite
$\lambda = \alpha \pm i\beta, \ \alpha < 0$		
$\lambda_1 = \lambda_2 > 0$	Indefinite	unstable
$\lambda_1 = \lambda_2 < 0$	Indifinite	asymptotically stable

---

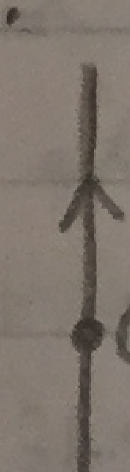
* Population Models *

① Rate of Growth is constantly k

$$\frac{dP}{dt} = kP$$

Critical points: $P = 0$

Phase Line:

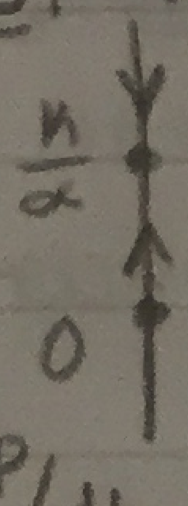


② Rate of Growth Decreases as Population Increases

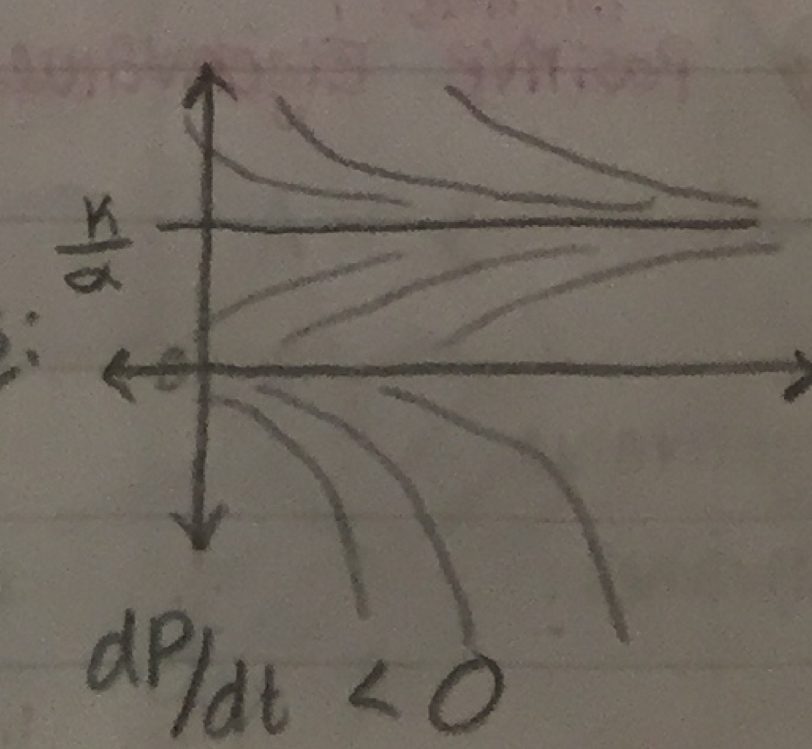
$$\frac{dP}{dt} = (k - \alpha x)x \quad \Big\} \text{ Logistic Model}$$

Critical Points: $x = 0$, $x = k/\alpha$

Phase Line:



Trajectories:



$\frac{k}{\alpha} > x > 0$ $\quad dP/dt > 0$, $\quad x > \frac{k}{\alpha}$ $\quad dP/dt < 0$

③ Competing Species:

x - Population of #1

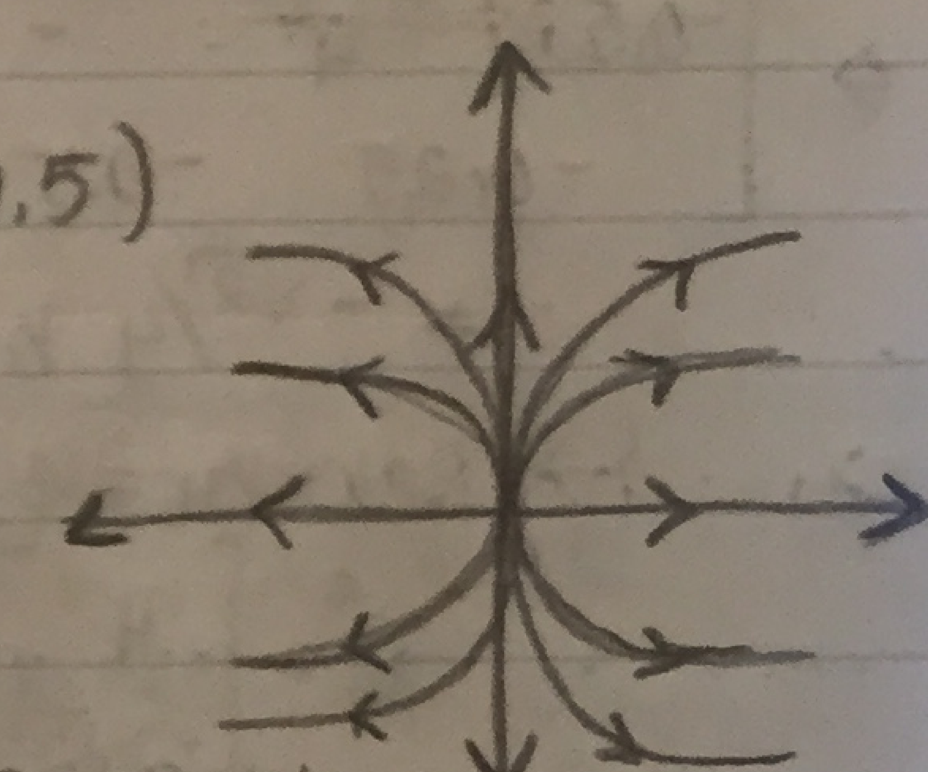
y - Population of #2

Rate of Growth Decreases if x increases or y increases for both #1 and #2

$$\begin{cases} dx/dt = (k_1 - \alpha_1 x - \beta_1 y)x \\ dy/dt = (k_2 - \alpha_2 y - \beta_2 y)y \end{cases}$$

Example: $$\begin{cases} dx/dt = x(1-x-y) \\ dy/dt = y(0.75 - y - 0.5x) \end{cases}$$

(i) Critical Points: $(0,0)$, $(1,0)$, $(0, 0.75)$, $(0.5, 0.5)$

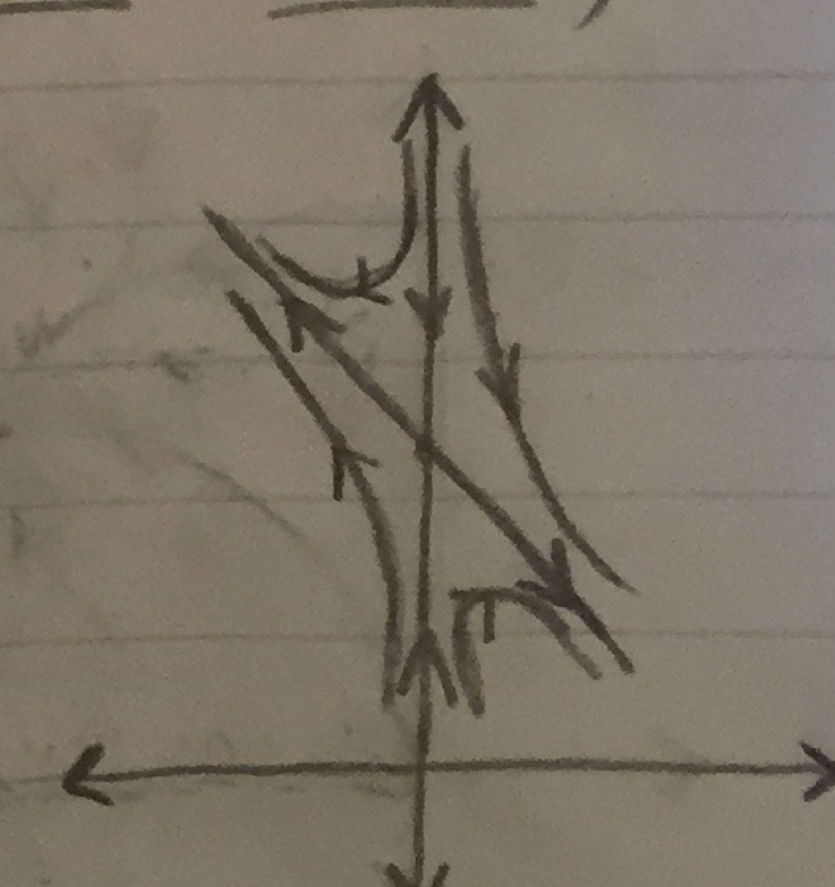


(ii) $$J(x,y) = \begin{bmatrix} 1-2x-y & -x \\ -0.5y & 0.75-2y-0.5x \end{bmatrix}$$

→ Near $(0,0)$: $\vec{u}' = \begin{bmatrix} 1 & 0 \\ 0 & 0.75 \end{bmatrix} \vec{u} \Rightarrow \lambda_1 = 1,\ \lambda_2 = 0.75$ Eigenvalues

$\Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ = Eigenvectors ⇒ Nodal Source (unstable)

→ Near $(0, 0.75)$: $\vec{u} = \begin{bmatrix} 0.25 & 0 \\ -0.375 & -0.75 \end{bmatrix} \vec{u}$



Eigenvalues: $\lambda_1 = 0.25$, $\lambda_2 = -0.75$

Eigenvectors:

$\vec{v}_1 = \begin{bmatrix} -8 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

⇒ $(0, 0.75)$ is a saddle point (unstable)

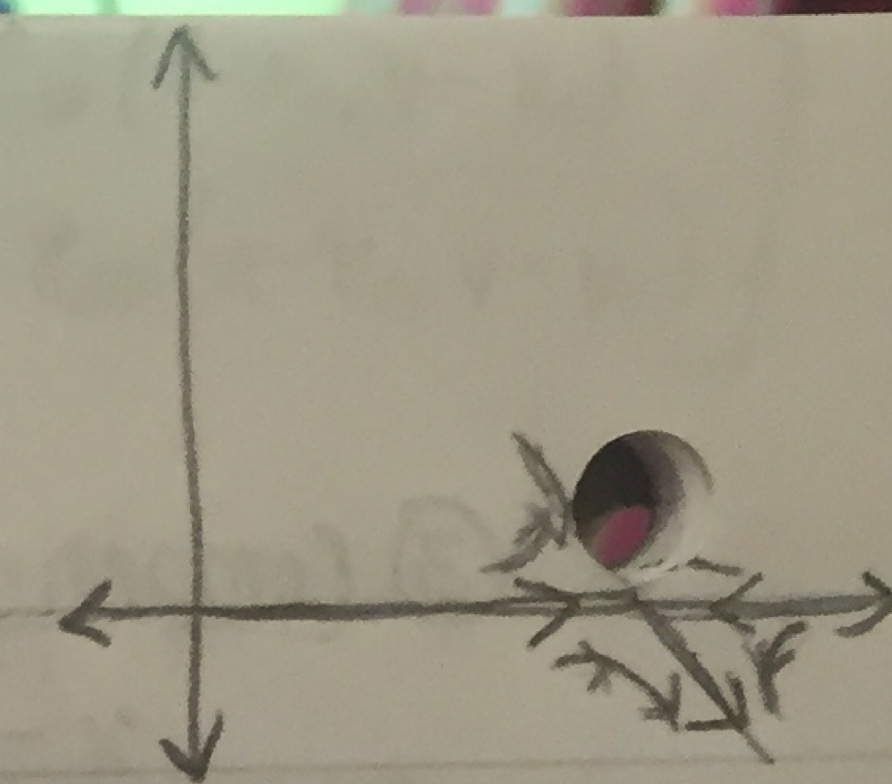
→ Near $(1,0)$: $\vec{u}' = \begin{bmatrix} -1 & -1 \\ 0 & 0.25 \end{bmatrix} \vec{u}$

Eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 0.25$

Eigenvectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

$\Rightarrow$ (1,0) is a saddle point (unstable)



→ Near (0.5, 0.5): $\vec{u}' = \begin{bmatrix} -0.5 & -0.5 \\ -0.25 & -0.5 \end{bmatrix} \vec{u}$

Eigenvalues: $\begin{vmatrix} -0.5-\lambda & -0.5 \\ -0.25 & -0.5-\lambda \end{vmatrix} = (-0.5-\lambda)^2 - 0.125 = 0$

$= \lambda^2 + \lambda + 0.25 - 0.125 = 0 \Rightarrow \lambda^2 + \lambda + 0.125 = 0$

$\Rightarrow \lambda = \dfrac{-1 \pm \sqrt{(1)^2 - 4(1)(0.125)}}{2(1)} = \dfrac{-2 \pm \sqrt{2}}{4}$

$\Rightarrow \left[\begin{array}{cc|c} -0.5 - \frac{-2+\sqrt{2}}{4} & -0.5 & 0 \\ -0.25 & -0.5 - \frac{-2+\sqrt{2}}{4} & 0 \end{array}\right] = \left[\begin{array}{cc|c} \frac{-\sqrt{2}}{4} & -0.5 & 0 \\ -0.25 & \frac{-\sqrt{2}}{4} & 0 \end{array}\right]$

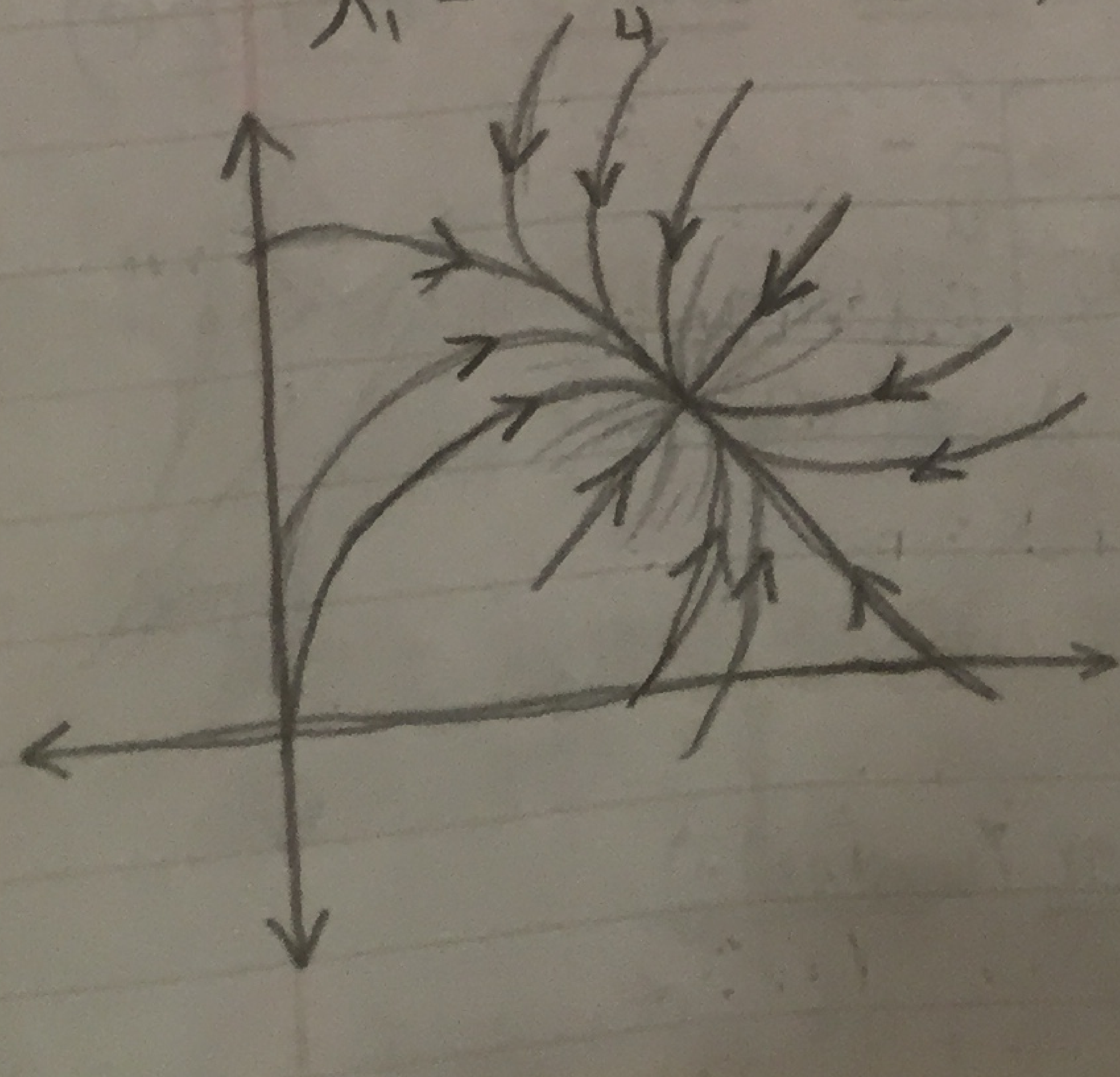
$\Rightarrow -\sqrt{2}/4 \, k_1 - 0.5 k_2 = 0$

↳ Set $k_1 = 4$, then $-\sqrt{2} - 0.5 k_2 = 0 \Rightarrow k_2 = -2\sqrt{2}$

$$\vec{v}_1 = \begin{bmatrix} 4 \\ -2\sqrt{2} \end{bmatrix} \approx \begin{bmatrix} 4 \\ -3.439 \end{bmatrix}$$

$\Rightarrow \left[\begin{array}{cc|c} -0.5 - \frac{-2-\sqrt{2}}{4} & -0.5 & 0 \\ -0.25 & -0.5 - \frac{-2-\sqrt{2}}{4} & 0 \end{array}\right] \Rightarrow \vec{v}_2 = \begin{bmatrix} 4 \\ 2\sqrt{2} \end{bmatrix}$

$\lambda_1 = \dfrac{-2\sqrt{2}}{4} < 0, \quad \lambda_2 = \dfrac{-2\sqrt{2}}{4} < 0$



Nodal Sink (Stable)